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Graded Majorana spinors

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Abstract

In many mathematical and physical contexts, spinors are treated as Grassmann odd valued fields. We show that it is possible to extend the classification of reality conditions on such spinors by a new type of Majorana condition. In order to define this *graded Majorana* condition we make use of pseudo-conjugation, a rather unfamiliar extension of complex conjugation to supernumbers. Like the symplectic Majorana condition, the graded Majorana condition may be imposed, for example, in spacetimes in which the standard Majorana condition is inconsistent. However, in contrast to the symplectic condition, which requires duplicating the number of spinor fields, the graded condition can be imposed on a *single* Dirac spinor. We illustrate how graded Majorana spinors can be applied to supersymmetry by constructing a globally supersymmetric field theory in three-dimensional Euclidean space, an example of a spacetime where standard Majorana spinors do not exist.

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1. Introduction

One of the key ingredients to a deep understanding of the mathematical concept of spinor fields has been the complete classification of all possible types of reality conditions that can be imposed on spinors in a given spacetime. If spinors are treated as ordinary fields, this classification of possible reality conditions, normally referred to as *Majorana* conditions, has been given in [1]. However, though this classification of Majorana conditions nicely extends to spinors treated as Grassmann odd valued fields, as is the case for example in supersymmetric theories, it turns out not to be complete. To see this, first note that the components of such Grassmann odd valued spinor fields are given by anticommuting supernumbers. Since a Majorana condition relates a spinor to its complex conjugate, extending the notion of a Majorana condition to such anticommuting spinor fields implies that one first has to extend

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the notion of complex conjugation to supernumbers. There is, however, an ambiguity in defining this extension, leading to at least two inequivalent notions of complex conjugation of supernumbers. These we will refer to as standard complex conjugation [2] and *pseudo-conjugation* [3], respectively. While standard complex conjugation essentially leads to the classification of Majorana conditions as given in [1], we show that pseudo-conjugation makes it possible to define a genuinely new type of Majorana spinor, which we will refer to as *graded Majorana*.

It should be pointed out that the existence of such reality conditions in the special case of four-dimensional Euclidean space has already been discussed in [4–6]. In this paper we will show how this special case is part of the wider and more general scheme of graded Majorana spinors which, as we shall see, are entirely complementary to standard Majorana spinors.

2. Pseudo-conjugation

Let us first briefly comment on the properties of standard complex conjugation and pseudo-conjugation, respectively. While the operation of standard complex conjugation on supernumbers is an involution, pseudo-conjugation in contrast is a *graded* involution. Denoting the operation of standard complex conjugation by $*$ and pseudo-conjugation by \diamond we thus have

$$z^{**} = z, \quad z^{\diamond\diamond} = (-1)^{\epsilon_z} z. \quad (2.1)$$

Here $\epsilon_z = 0$ if z is an even (commuting) supernumber, and $\epsilon_z = 1$ if z is odd (anticommuting). It is this property of pseudo-conjugation which will enable us later to define a new kind of Majorana spinor. Additionally, standard complex conjugation and pseudo-conjugation, respectively, satisfy the properties

$$(z + w)^* = z^* + w^*, \quad (z + w)^\diamond = z^\diamond + w^\diamond, \quad (2.2a)$$

$$(zw)^* = w^* z^*, \quad (zw)^\diamond = z^\diamond w^\diamond. \quad (2.2b)$$

Note that both types of conjugation reduce to ordinary complex conjugation on ordinary numbers.

A general supernumber can be expanded in the generators $\zeta_i, i = 1, \dots, N$, of a Grassmann algebra as

$$z = z_0 + z_i \zeta_i + z_{ij} \zeta_i \zeta_j + z_{ijk} \zeta_i \zeta_j \zeta_k + \dots \quad (2.3)$$

Here the coefficients z_0, z_i, \dots are ordinary complex numbers. With respect to standard complex conjugation the generators will be taken to be real, i.e., $\zeta_i^* = \zeta_i$. However, imposing a similar reality condition on the generators using pseudo-conjugation will be inconsistent with equation (2.1). Instead, without loss of generality, we will impose

$$\zeta_{2i}^\diamond = \zeta_{2i-1}, \quad \zeta_{2i-1}^\diamond = -\zeta_{2i}. \quad (2.4)$$

This requires the number N of Grassmann generators to be even or—as one normally considers in the context of supersymmetric theories—infinite. Note that $\zeta_i^{*\diamond} = \zeta_i^{\diamond*}$, from which it follows that standard complex conjugation commutes with pseudo-conjugation on arbitrary supernumbers.

As we shall see, it will be convenient to split the supernumber z into a sum of two parts

$$z = z_1 + z_2, \quad (2.5a)$$

$$z_1 = \frac{1}{2}(z + z^{*\diamond}), \quad z_2 = \frac{1}{2}(z - z^{*\diamond}). \quad (2.5b)$$

Using this splitting we define an invertible map f on even supernumbers z

$$f : z \rightarrow \tilde{z} = z_1 + iz_2, \tag{2.6a}$$

$$f^{-1} : \tilde{z} \rightarrow z = \tilde{z}_1 - i\tilde{z}_2, \tag{2.6b}$$

with $\tilde{z}_{1,2}$ defined analogously to $z_{1,2}$ in equation (2.5b). This map satisfies the property

$$f(z^\diamond) = f(z)^*, \tag{2.7}$$

which can be shown using the fact that $z_1^\diamond = z_1^*$ and $z_2^\diamond = -z_2^*$. It follows that imposing a pseudo-reality condition $z = z^\diamond$ on an arbitrary even supernumber z is equivalent to imposing the standard reality condition $f(z) = f(z)^*$ on the supernumber $f(z) = \tilde{z}$.

In section 4 we will consider how pseudo-conjugation may be used to impose reality conditions on spinors, the components of which are taken to be anticommuting supernumbers. However, we first need to recall some results about Clifford algebras, as discussed in [1].

3. Clifford algebras in d -dimensions

The Clifford algebra in d spacetime dimensions is given by

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= 2\eta^{\mu\nu} \mathbb{1}, \\ \eta^{\mu\nu} &= \text{diag}(\underbrace{+ \dots +}_t \underbrace{- \dots -}_s), \end{aligned} \tag{3.1}$$

with $d = t + s$. The γ^μ are represented by $2^{\lfloor d/2 \rfloor} \times 2^{\lfloor d/2 \rfloor}$ matrices, which may be chosen such that

$$\gamma^{\mu\dagger} = \gamma^\mu, \quad \mu = 1, \dots, t, \tag{3.2a}$$

$$\gamma^{\mu\dagger} = -\gamma^\mu, \quad \mu = t + 1, \dots, d. \tag{3.2b}$$

Defining $A = \gamma^1 \dots \gamma^t$ we then have

$$\gamma^{\mu\dagger} = -(-1)^t A \gamma^\mu A^{-1}. \tag{3.3}$$

In even dimensions we can introduce the matrix

$$\Gamma_5 = (-1)^{(t-s)/4} \gamma^1 \dots \gamma^d, \tag{3.4}$$

which satisfies $(\Gamma_5)^2 = 1$ and is, up to proportionality, the unique matrix which anticommutes with all γ^μ , $\mu = 1, \dots, d$. As $\pm\gamma^{\mu*}$ form an equivalent representation of the Clifford algebra, there exists an invertible matrix B such that

$$\gamma^{\mu*} = \eta B \gamma^\mu B^{-1}, \quad \eta = \pm 1, \tag{3.5}$$

where η can be shown to depend on the signature of the metric, see table 1. Note that in even dimensions, where $t - s$ will also be even, we always have a choice of $\eta = \pm 1$, whereas in odd dimensions η is fixed. B is unitary and satisfies the condition

$$B^* B = \epsilon \mathbb{1}, \quad \epsilon = \pm 1, \tag{3.6}$$

where ϵ depends on the signature of the metric as well as on the value of η as displayed in table 1. Note that B is only defined up to an overall phase.

The charge conjugation matrix C is defined by

$$C = B^T A. \tag{3.7}$$

Using the properties of A and B one finds that $C^\dagger C = \mathbb{1}$ and

$$\gamma^{\mu T} = (-1)^{t+1} \eta C \gamma^\mu C^{-1}, \tag{3.8}$$

$$C^T = \epsilon \eta^t (-1)^{t(t-1)/2} C. \tag{3.9}$$

Table 1. Possible values of ϵ and η in all signatures.

$t - s \bmod 8$	0, 1, 2	0, 6, 7	4, 5, 6	2, 3, 4
ϵ	+1	+1	-1	-1
η	+1	-1	+1	-1
Spinor type	M'	M	gM'	gM

The last two equations can be combined to give

$$(\gamma^\mu C^{-1})^T = (-1)^{t+1+t(t-1)/2} \epsilon \eta^{t+1} (\gamma^\mu C^{-1}). \tag{3.10}$$

Additionally we have that

$$(\gamma^\mu C^{-1})^* = \eta^{t+1} B \gamma^\mu C^{-1} B^T. \tag{3.11}$$

These two relations will be important when considering super-Poincaré algebras in different signatures, see section 5.

In even dimensions, as there is a choice of $\eta = \pm 1$, let us define B_\pm such that

$$\gamma^{\mu*} = \eta_\pm B_\pm \gamma^\mu B_\pm^{-1}, \quad B_\pm^* B_\pm = \epsilon_\pm \mathbb{1}. \tag{3.12}$$

Here $\eta_\pm = \pm 1$ and ϵ_\pm is the value of ϵ corresponding to η_\pm in a given signature. Correspondingly we define $C_\pm = B_\pm^T A$.

Interestingly B_+ and B_- are related by

$$B_+ = \lambda B_- \Gamma_5, \tag{3.13}$$

where λ is an arbitrary phase factor. This relation seems to have been overlooked in the literature. To prove equation (3.13) note that

$$B_-^{-1} B_+ \gamma^\mu B_+^{-1} B_- = B_-^{-1} \gamma^{\mu*} B_- = -\gamma^\mu, \tag{3.14}$$

hence $B_-^{-1} B_+$ anticommutes with all the gamma matrices and as such must be proportional to Γ_5 . Unitarity of both B_\pm and Γ_5 restricts λ such that $|\lambda|^2 = 1$.

Note that using the relation between B_+ and B_- , equation (3.13), we find $\epsilon_+ \mathbb{1} = B_+^* B_+ = |\lambda|^2 (-1)^{(t-s)/2} \epsilon_- \mathbb{1}$, and hence we see that

$$\epsilon_+ = (-1)^{(t-s)/2} \epsilon_-, \tag{3.15}$$

which is in agreement with table 1.

4. Reality conditions on spinors

In many contexts spinors are treated as Grassmann odd valued fields, i.e. the $2^{\lfloor d/2 \rfloor}$ components of a general Dirac spinor are given by anticommuting complex supernumbers. Depending on the signature of the spacetime under consideration such spinors can be constrained by reality conditions that are both consistent with the Dirac equation and Lorentz covariant. Reality conditions that satisfy these requirements are normally referred to as *Majorana* conditions. Conventionally, only standard complex conjugation of supernumbers has been used to impose such Majorana conditions. In this section we shall show how, by using pseudo-conjugation of supernumbers, a genuinely new type of Majorana condition can be defined.

4.1. Standard and symplectic Majorana conditions

Let us first consider signatures in which there exists a matrix B for which $\epsilon = +1$, i.e. $B^* B = \mathbb{1}$, see table 1. We may use this matrix B to impose the standard *Majorana* condition

$$\psi = B^{-1} \psi^*. \tag{4.1}$$

Note that imposing such a condition will not be consistent if $\epsilon = -1$ since $\psi = B^{-1}(B^{-1}\psi^*)^* = (B^*B)^{-1}\psi = \epsilon\psi$.

In those signatures where there are only matrices B for which $\epsilon = -1$ one normally introduces a pair (or more generally an even number) of Dirac spinors $\psi^{(i)}, i = 1, 2$, and imposes the *symplectic* Majorana condition

$$\psi^{(i)} = \epsilon^{ij} B^{-1}(\psi^{(j)})^* \quad \text{for} \quad B^*B = -\mathbb{1}, \tag{4.2}$$

where $\epsilon^{ij} = -\epsilon^{ji}$ with $\epsilon^{12} = +1$. This condition reduces the degrees of freedom of the pair of spinors down to that of a single spinor with no reality condition imposed. Therefore, since a second spinor is initially introduced in order to impose the symplectic Majorana condition, the number of degrees of freedom is not in effect reduced.

4.2. Graded Majorana conditions

We shall now show that in signatures in which there exists a matrix B for which $\epsilon = -1$, i.e. $B^*B = -\mathbb{1}$, we can—by making use of pseudo-conjugation—define an alternative Majorana condition that, unlike the symplectic one, does not require duplicating the number of fields, but instead can be imposed on a *single* spinor. We propose the condition

$$\psi = B^{-1}\psi^\diamond. \tag{4.3}$$

Now, since the components of ψ are anticommuting supernumbers, we have from equation (2.1) that $\psi^\diamond = -\psi$, hence $\psi = B^{-1}(B^{-1}\psi^\diamond)^\diamond = (B^*B)^{-1}\psi^\diamond = -\epsilon\psi$ and so the equation is consistent for $\epsilon = -1$. Note that here we have used $B^\diamond = B^*$ since B is a matrix of ordinary complex numbers. As pseudo-conjugation is a graded involution we will refer to spinors satisfying equation (4.3) as *graded Majorana* spinors.

To be complete we also note here that, in those signatures for which there exists a matrix B for which $\epsilon = +1$, pseudo-conjugation may be used to define a *graded symplectic* Majorana condition

$$\psi^{(i)} = \epsilon^{ij} B^{-1}(\psi^{(j)})^\diamond \quad \text{for} \quad B^*B = +\mathbb{1}. \tag{4.4}$$

Equations (4.3), (4.4) thus constitute the graded counterparts of equations (4.1), (4.2) and highlight how graded Majorana conditions should be treated on an equal footing with the standard Majorana conditions.

In the next section we will show how reality conditions using standard complex conjugation and pseudo-conjugation, respectively, can be thought of as equivalent in terms of the number of constraints they impose on a spinor.

4.3. Equivalence of reality conditions

Just as the standard Majorana condition of equation (4.1) is covariant under Lorentz transformations so, too, is the graded Majorana condition of equation (4.3). For the purpose of analysing the number of constraints, however, we shall also consider more general reality conditions that may not necessarily be so. Let us introduce $2^{\lfloor d/2 \rfloor} \times 2^{\lfloor d/2 \rfloor}$ matrices M and N satisfying $M^*M = +\mathbb{1}$ and $N^*N = -\mathbb{1}$, respectively (where we require $d > 1$ for the matrix N to exist). Then consider reality conditions of the form $\psi = M^{-1}\psi^*$ and $\psi = N^{-1}\psi^\diamond$, encompassing the standard and graded Majorana conditions, respectively. In particular these conditions shall be replaced with the corresponding Majorana conditions, equations (4.1), (4.3), as long as the appropriate matrices B exist.

In order to show that the number of constraints imposed on a spinor is the same for both $\psi = M^{-1}\psi^*$ and $\psi = N^{-1}\psi^\diamond$ we will use an argument analogous to that for an even

supernumber as discussed in section 2. Consider the split of equations (2.5a), (2.5b) applied to each of the components of the spinor ψ , resulting in

$$\psi = \psi_1 + \psi_2, \tag{4.5a}$$

$$\psi_1 = \frac{1}{2}(\psi + \psi^{*\diamond}), \quad \psi_2 = \frac{1}{2}(\psi - \psi^{*\diamond}). \tag{4.5b}$$

Using the fact that $\psi_1^* = \psi_2^\diamond$ and $\psi_2^* = -\psi_1^\diamond$ it is easily seen that the following two equivalences hold:

$$\psi = M^{-1}\psi^* \iff \begin{cases} \psi_1 = M^{-1}\psi_2^\diamond \\ \psi_2 = -M^{-1}\psi_1^\diamond \end{cases} \tag{4.6}$$

and

$$\psi = N^{-1}\psi^\diamond \iff \begin{cases} \psi_1 = -N^{-1}\psi_2^* \\ \psi_2 = N^{-1}\psi_1^*. \end{cases} \tag{4.7}$$

In those signatures where there exists a matrix B such that $B^*B = -\mathbb{1}$, equation (4.7) shows how a graded Majorana condition imposed on the spinor ψ can be restated as a symplectic Majorana condition imposed on the *split* fields $\psi_{1,2}$ of equation (4.5a). Note, however, that the symplectic Majorana condition is being imposed on the *internal* supernumber structure of a *single* spinor. Conversely, in those signatures where there exists a matrix B such that $B^*B = \mathbb{1}$, we see from equation (4.6) that the standard Majorana condition is equivalent to a graded symplectic Majorana condition being imposed on the split fields $\psi_{1,2}$. Also in this case the symplectic condition is imposed on the internal supernumber structure of a single spinor.

Let us now define the quantity

$$\tilde{\psi} = \mu^*\psi_1 + \mu M^*N\psi_2, \tag{4.8}$$

where μ is some non-zero, ordinary complex constant. The relationship of equation (4.8) may be inverted to give ψ in terms of $\tilde{\psi}$. To see this note that if we split $\tilde{\psi}$ as in equations (4.5a), (4.5b) we have

$$\tilde{\psi}_1 = \frac{1}{2}((\mu^* + \mu M^*N)\psi_1 - (\mu^* - \mu M^*N)\psi_2), \tag{4.9a}$$

$$\tilde{\psi}_2 = \frac{1}{2}((\mu^* - \mu M^*N)\psi_1 + (\mu^* + \mu M^*N)\psi_2), \tag{4.9b}$$

where we have used that $\psi_1^* = \psi_2^\diamond$ and $\psi_2^* = -\psi_1^\diamond$. We then find

$$\psi_1 = \Delta^{-1}((\mu M^*N + \mu^*)\tilde{\psi}_1 - (\mu M^*N - \mu^*)\tilde{\psi}_2), \tag{4.10a}$$

$$\psi_2 = \Delta^{-1}((\mu M^*N - \mu^*)\tilde{\psi}_1 + (\mu M^*N + \mu^*)\tilde{\psi}_2), \tag{4.10b}$$

where $\Delta \equiv (\mu^*)^2\mathbb{1} + \mu^2(M^*N)^2$. For Δ to be invertible we must choose μ such that $\pm i\mu^*/\mu$ is not an eigenvalue of M^*N , which is always possible. Hence, we find for ψ in terms of $\tilde{\psi}$

$$\psi = 2\Delta^{-1}(\mu M^*N\tilde{\psi}_1 + \mu^*\tilde{\psi}_2). \tag{4.11}$$

We can now show that a reality condition on ψ using pseudo-conjugation is, in terms of the number of constraints imposed, equivalent to a reality condition on $\tilde{\psi}$ using standard complex conjugation. From equations (4.9a–4.10b) and the fact that $\psi_1^* = \psi_2^\diamond$ and $\psi_2^* = -\psi_1^\diamond$ we have

$$\left. \begin{matrix} \psi_1 = -N^{-1}\psi_2^* \\ \psi_2 = N^{-1}\psi_1^* \end{matrix} \right\} \iff \left\{ \begin{matrix} \tilde{\psi}_1 = M^{-1}\tilde{\psi}_2^\diamond \\ \tilde{\psi}_2 = -M^{-1}\tilde{\psi}_1^\diamond. \end{matrix} \right. \tag{4.12}$$

Now, combining equations (4.6), (4.7) with equation (4.12) we find that

$$\psi = N^{-1}\psi^\diamond \iff \tilde{\psi} = M^{-1}\tilde{\psi}^*. \tag{4.13}$$

As there exists an invertible map between ψ and $\tilde{\psi}$, this proves that a reality condition using pseudo-conjugation imposes the same number of constraints as does a reality condition using standard complex conjugation².

4.4. Dirac equation and spinor actions

If $\eta(-1)^t = -1$, see table 1, the Dirac equation for the corresponding Majorana spinors is not consistent with a mass term [1]. It will therefore be necessary to distinguish between the Majorana conditions corresponding to the two possible cases $\eta = \pm 1$. Consider first the standard Majorana condition. If $\eta = -1$ the spinor will simply be referred to as Majorana (M). If however $\eta = +1$ the spinor will be called *pseudo-Majorana* (M'). Similarly, for the graded Majorana condition, the spinor will be called graded Majorana (gM) if $\eta = -1$ and *pseudo-graded Majorana* (gM') if $\eta = +1$. See table 1 for a summary. Consequently, in Minkowski space, pseudo-Majorana spinors must be massless to be consistent with the Dirac equation and the same is true for pseudo-graded Majorana spinors.

Now one should note that the Dirac equation for Majorana spinors cannot always be derived from an action. Whether or not this is possible depends on the respective Majorana condition used and on the symmetry properties of $C\gamma_\mu$ and C . The Lagrangian for both standard and graded Majorana spinors will be of the form³

$$\mathcal{L} = \psi^T C (i^t \gamma^\mu \partial_\mu - m) \psi. \tag{4.14}$$

In the case of standard Majorana spinors one easily finds that for the action to be non-vanishing one has to require $C\gamma_\mu$ to be symmetric, and, if massive, we further require the charge conjugation matrix C to be antisymmetric [7]. In the case of graded Majorana spinors the same conditions apply. Note that in Minkowski spacetimes we have $(C\gamma_\mu)^T = \epsilon C\gamma_\mu$, therefore an action involving graded Majorana spinors ($\epsilon = -1$) will vanish. In Euclidean or other signatures, however, this need not be the case. In Euclidean signatures, for example, an action involving standard Majorana spinors is non-vanishing only if $d = 0, 1, 2 \pmod 8$, whereas an action involving graded Majorana spinors is non-vanishing only if $d = 2, 3, 4 \pmod 8$.

If instead we consider parity violating Lagrangians of the form

$$\mathcal{L} = \psi^T C \Gamma_5 (i^t \gamma^\mu \partial_\mu - m) \psi \tag{4.15}$$

we require $C\Gamma_5\gamma^\mu$ to be symmetric and, in the case of massive spinors, we also require $C\Gamma_5$ to be antisymmetric (note that d must be even for Γ_5 to exist). Now in Minkowski spacetimes we have $(C\Gamma_5\gamma^\mu)^T = -\epsilon(-1)^{d/2}C\Gamma_5\gamma^\mu$. Therefore such an action involving graded Majorana spinors will be non-vanishing in Minkowski spacetimes only if $d = 0 \pmod 4$, whereas in the case of standard Majorana spinors we require $d = 2 \pmod 4$.

Finally let us consider the Dirac action for a pair of symplectic Majorana spinors. In this case we have

$$\mathcal{L} = \psi^{(i)T} C \epsilon^{ij} (i^t \gamma^\mu \partial_\mu - m) \psi^{(j)}. \tag{4.16}$$

For the action to be non-vanishing we require that $C\gamma^\mu$ be antisymmetric, and in the massive case we additionally require C to be symmetric.

² Note that in most cases only one of ψ or $\tilde{\psi}$ can be chosen to have the correct transformation properties under the Lorentz group in order to be regarded as a spinor. In the cases where $t - s = 2 \pmod 4$, both ψ and $\tilde{\psi}$ can be chosen to transform as spinors.

³ The appearance of the factor i^t is due to the fact that the mass term and the derivative term should have the same reality properties under (pseudo-) conjugation.

4.5. Standard and graded Majorana–Weyl conditions

Note that in even dimensions, where we have a choice of matrices B_{\pm} for $\eta = \pm 1$, it is possible to simultaneously impose the two corresponding reality conditions. Such spinors will be massless due to the fact that one of the Majorana conditions imposed will necessarily be inconsistent with a mass term. There are four possible cases which we shall analyse separately.

If $t - s = 0 \pmod 8$ we can impose both M and M' conditions, giving

$$\psi = B_-^{-1}\psi^* = B_+^{-1}\psi^*. \quad (4.17)$$

Using equation (3.13) we see that a consequence of these two conditions is that

$$\psi = \lambda \Gamma_5 \psi. \quad (4.18)$$

This equation will only be consistent if $\lambda = \pm 1$, in which case equation (4.18) is seen to be the Weyl condition for a spinor with helicity λ . Note that the Weyl condition can be imposed on spinors in any even-dimensional spacetime. Here, however, the spinors are also Majorana and we see that consistently imposing both Majorana conditions has naturally given a *Majorana–Weyl (MW)* condition. Note that the helicity of the resulting Majorana–Weyl spinor depends on the value of λ and as such on the relative phase chosen between the matrices B_+ and B_- in equation (3.13)⁴.

If $t - s = 4 \pmod 8$ we can impose both gM and gM' conditions

$$\psi = B_-^{-1}\psi^{\diamond} = B_+^{-1}\psi^{\diamond}. \quad (4.19)$$

Again we have as a consequence of these equations that ψ must satisfy the Weyl condition, equation (4.18), with helicity $\lambda = \pm 1$ for consistency. We refer to such spinors as *graded Majorana–Weyl (gMW)*.

If $t - s = 2 \pmod 8$ we can impose both gM and M' conditions

$$\psi = B_-^{-1}\psi^{\diamond} = B_+^{-1}\psi^*. \quad (4.20)$$

The Weyl condition, equation (4.18), is no longer satisfied due to the mixed nature of the Majorana conditions. Instead, a consequence of these two conditions is

$$\psi = \lambda \Gamma_5 \psi^{*\diamond}, \quad (4.21)$$

where for consistency we must have $\lambda = \pm i$. Note that, although ψ is not a true Weyl spinor, if we split $\psi = \psi_1 + \psi_2$ as in equations (4.5a), (4.5b) then the combinations $\psi_1 \pm i\psi_2$ are Weyl. However, the physical interpretation of the condition in equation (4.21) remains unclear.

If $t - s = 6 \pmod 8$ we have both M and gM' conditions. This case is very similar to $t - s = 2 \pmod 8$.

Table 2 summarizes which reality conditions may be imposed in each of the most interesting spacetimes.

4.6. Four-dimensional Euclidean space

It is worth mentioning here that when working in even dimensions it is common to use the Weyl representation for spinors. The Weyl representation can be defined in full generality for arbitrary signature in any even-dimensional spacetime; however it is perhaps most familiar in four-dimensional Minkowski space where the use of two-component spinors with dotted and undotted indices is quite standard. Here, however, we shall briefly discuss the case of four-dimensional Euclidean space, demonstrating how the reality conditions imposed in [4–6] fit into the general scheme of graded Majorana spinors.

⁴ Remember that the matrices B_+ and B_- are defined up to an overall phase only.

Table 2. Possible types of maximal reality conditions in different spacetimes.

$t \backslash d$	1	2	3	4	5	6	7	8	9	10	11	12
0	□	▼	■'	◆	■	▲	□'	◇	□	▼	■'	◆
1	□'	◇	□	▼	■'	◆	■	▲	□'	◇	□	▼
2		▲	□'	◇	□	▼	■'	◆	■	▲	□'	◇
$t = d$	□'	▲	■	◆	■'	▼	□	◇	□'	▲	■	◆

□ M □' M' ◇ MW ▲ gM & M'
 ■ gM ■' gM' ◆ gMW ▼ M & gM'

The four-dimensional Euclidean gamma matrices are taken to be

$$\gamma^i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{4.22}$$

Here $i = 1, 2, 3$ and σ_i are the standard Pauli matrices. We choose the matrices B_{\pm} in this representation to be

$$B_{\pm} = \begin{pmatrix} -\varepsilon & 0 \\ 0 & \mp\varepsilon \end{pmatrix}, \tag{4.23}$$

where $\varepsilon = i\sigma_2$. We see from the form of Γ_5 that the four-component Dirac spinor decomposes into left- and right-handed two-component spinors, ϕ and χ , as

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \tag{4.24}$$

The graded Majorana conditions, $\psi = B_{\pm}^{-1}\psi^{\diamond}$, are then simply

$$\phi = \varepsilon\phi^{\diamond}, \quad \chi = \pm\varepsilon\chi^{\diamond}. \tag{4.25}$$

Note that with this choice of the matrices B_{\pm} imposing both graded Majorana conditions implies $\chi = 0$ and hence the resulting spinor will be a left-handed graded Majorana-Weyl spinor. If we had chosen the opposite relative sign between B_+ and B_- the resulting spinor would have been right-handed.

Introducing indices $a, b, \dots = 1, 2$ for left-handed spinors, and $a', b', \dots = 1, 2$ for right-handed spinors, we find for equation (4.25) upon displaying the indices explicitly

$$\phi_a = \varepsilon_{ab}(\phi_b)^{\diamond}, \quad \chi_{a'} = \pm\varepsilon_{a'b'}(\chi_{b'})^{\diamond}. \tag{4.26}$$

These expressions may be compared to the reality conditions imposed in [4–6]. Note that in this signature pseudo-conjugation does not change the index type from primed to unprimed. This is due to the fact that the left-handed and right-handed components of $\text{Spin}(4)$ do not mix under conjugation [4, 6], a situation which can be contrasted with, for example, four-dimensional Minkowski space where conjugation acts to interchange the left-handed and right-handed components of $\text{Spin}(1, 3)$ [8].

5. Applications to supersymmetry

5.1. Real forms of the super-Poincaré algebra

We shall now investigate how these new reality conditions can be imposed to give real forms of super Lie algebras, which will subsequently allow the derivation of supersymmetric field theories involving graded Majorana spinors.

Let us define the graded commutator $[K, L] = KL - (-1)^{\epsilon_K \epsilon_L} LK$, where $\epsilon_K = 0$ if K is even and $\epsilon_K = 1$ if K is odd (and similarly for L). The generators of the general $\mathcal{N} = 1$ super-Poincaré algebra satisfy

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}, \quad (5.1a)$$

$$[M_{\mu\nu}, P_\rho] = \eta_{\rho\nu} P_\mu - \eta_{\rho\mu} P_\nu, \quad (5.1b)$$

$$[M_{\mu\nu}, Q_\alpha] = -(\sigma_{\mu\nu})_\alpha^\beta Q_\beta, \quad (5.1c)$$

$$[Q_\alpha, Q_\beta] = 2k(\gamma^\mu C^{-1})_{\alpha\beta} P_\mu, \quad (5.1d)$$

where all other commutators vanish. Here the even generators $M_{\mu\nu}$ and P_μ , generating rotations and translations, respectively, form the Poincaré subalgebra, and Q_α are the odd supersymmetry generators forming a $2^{\lfloor d/2 \rfloor}$ component spinor. We choose $(\gamma^\mu)_\alpha^\beta$ to correspond to the components of the gamma matrices and $C^{\alpha\beta}$ to correspond to the components of the charge conjugation matrix C . Note that with these index conventions $C^{-1} = ((C^{-1})_{\alpha\beta})$. We have $\sigma^{\mu\nu} = (1/4)(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$ and k appearing in equation (5.1d) is a constant phase factor which will be determined when considering a specific real form of the algebra. Note that if there is no matrix C available such that $\gamma^\mu C^{-1}$ is symmetric, see equation (3.10), such an $\mathcal{N} = 1$ algebra does not exist. One may, however, instead consider an $\mathcal{N} \geq 2$ algebra.

The general element of the super-Poincaré algebra is given by

$$X = \omega^{\mu\nu} M_{\mu\nu} + x^\mu P_\mu + \theta^\alpha Q_\alpha. \quad (5.2)$$

Here $\omega^{\mu\nu}$, x^μ are even supernumbers and θ^α are odd supernumbers forming a Dirac conjugate spinor. In order to define a real form of the algebra these coefficients must be constrained by reality conditions such that the algebra still closes. This can be achieved by using standard complex conjugation or pseudo-conjugation, respectively.

To impose reality conditions using pseudo-conjugation we require that there exists a matrix⁵ $B = (B^{\alpha\beta})$ for which $\epsilon = -1$. A consistent choice of reality conditions is then given by

$$(\omega^{\mu\nu})^\diamond = \omega^{\mu\nu}, \quad (x^\mu)^\diamond = x^\mu, \quad (\theta^\alpha)^\diamond B^{\alpha\beta} = \theta^\beta. \quad (5.3)$$

The condition on θ^α can be viewed as a graded Majorana condition imposed on a Dirac conjugate spinor. Note that we consider the (pseudo-)conjugate of a quantity with an upstairs spinor index to have a downstairs index, and vice versa. It is easily seen that the Poincaré subalgebra of equations (5.1a), (5.1b) closes under the reality conditions of equation (5.3). To show closure of the full super-Poincaré algebra let us first consider equation (5.1c). We have

$$[\omega^{\mu\nu} M_{\mu\nu}, \theta^\alpha Q_\alpha] = -\omega^{\mu\nu} \theta^\alpha (\sigma_{\mu\nu})_\alpha^\beta Q_\beta. \quad (5.4)$$

For consistency with equation (5.3) the coefficient of Q_β on the right-hand side of the above equation must satisfy

$$-\omega^{\mu\nu} \theta^\alpha (\sigma_{\mu\nu})_\alpha^\beta = -(\omega^{\mu\nu} \theta^\alpha (\sigma_{\mu\nu})_\alpha^\gamma)^\diamond B^{\gamma\beta}, \quad (5.5)$$

which is easily checked using the fact that $(\sigma_{\mu\nu})^* = B \sigma_{\mu\nu} B^{-1}$. Further we see from this that the condition $(\theta^\alpha)^\diamond B^{\alpha\beta} = \theta^\beta$ is Lorentz covariant. Finally let us consider equation (5.1d). We have

$$[\theta^\alpha Q_\alpha, \tilde{\theta}^\beta Q_\beta] = -2k \theta^\alpha \tilde{\theta}^\beta (\gamma^\mu C^{-1})_{\alpha\beta} P_\mu. \quad (5.6)$$

⁵ Here we shall assume for simplicity that B and C are related by $C = B^T A$. In even dimensions there may occur more general situations which, using equation (3.13), can be treated similarly to this case.

For the algebra to close under the reality conditions, equation (5.3), the coefficient of P_μ on the right-hand side of the equation must be real with respect to pseudo-conjugation. Using equation (3.11) we find

$$\begin{aligned} (k\theta^\alpha \tilde{\theta}^\beta (\gamma^\mu C^{-1})_{\alpha\beta})^\diamond &= k^* (\theta^\alpha)^\diamond (\tilde{\theta}^\beta)^\diamond ((\gamma^\mu C^{-1})_{\alpha\beta})^* \\ &= k^* \eta^{t+1} (\theta^\alpha)^\diamond (\tilde{\theta}^\beta)^\diamond (B \gamma^\mu C^{-1} B^T)^{\alpha\beta} \\ &= k^* \eta^{t+1} (\theta^\alpha)^\diamond B^{\alpha\gamma} (\tilde{\theta}^\beta)^\diamond (B^T)^{\delta\beta} (\gamma^\mu C^{-1})_{\gamma\delta} \\ &= k^* \eta^{t+1} \theta^\gamma \tilde{\theta}^\delta (\gamma^\mu C^{-1})_{\gamma\delta}. \end{aligned} \tag{5.7}$$

Hence, provided we choose k such that $k = k^* \eta^{t+1}$, the algebra closes under the reality conditions, equation (5.3), which therefore give a real form of the algebra.

One can alternatively use standard complex conjugation in order to define a real form of the algebra equation (5.2). A consistent choice of reality conditions on the coefficients is, in this case, given by

$$(\omega^{\mu\nu})^* = \omega^{\mu\nu}, \quad (x^\mu)^* = x^\mu, \quad (\theta^\alpha)^* B^{\alpha\beta} = \theta^\beta, \tag{5.8}$$

provided, of course, B is now such that $\epsilon = +1$. That the super-Poincaré algebra also closes under these conditions can be proven analogously to the case of pseudo-conjugation. In this case however we find $k = -k^* \eta^{t+1}$.

In even dimensions we have the possibility of imposing two Majorana conditions on the coefficients θ^α . Due to the resulting Weyl condition if $t - s = 0, 4 \pmod 8$ we must, in these signatures, replace equation (5.1d) with

$$[Q_\alpha, Q_\beta] = 2k(\mathbb{1} + \lambda\Gamma_5)_{\alpha\gamma} (\gamma^\mu C^{-1})_{\gamma\beta} P_\mu, \tag{5.9}$$

which is possible provided that both $\Gamma_5 \gamma^\mu C^{-1}$ and $\gamma^\mu C^{-1}$ are symmetric (note that here C is a particular choice of $C_\pm = B_\pm^T A$). It is then possible to define a real form of the algebra by imposing MW or gMW conditions on the Dirac conjugate spinor (θ^α) with corresponding reality conditions on the $\omega^{\mu\nu}$ s and x^μ s. For example, let us consider $t - s = 4 \pmod 8$. The algebra will close if we impose the gMW condition

$$\theta^\alpha = (\theta^\beta)^\diamond (B_-)^{\beta\alpha} = (\theta^\beta)^\diamond (B_+)^{\beta\alpha} \tag{5.10}$$

along with the conditions $(\omega^{\mu\nu})^\diamond = \omega^{\mu\nu}$ and $(x^\mu)^\diamond = x^\mu$. If $t - s = 2, 6 \pmod 8$ we may consistently impose both a graded and a standard Majorana condition on the coefficients θ^α . However, the physical interpretation of such mixed reality conditions remains unclear.

5.2. Three-dimensional Euclidean field theory

In order to illustrate the applications of graded Majorana spinors to supersymmetric field theories, let us construct a simple example in three-dimensional Euclidean space (i.e., $t = 3, s = 0$). From table 1 we see that $\epsilon = -1$ and so no standard Majorana spinors exist. We choose the gamma matrices to be the standard Pauli matrices $\gamma^i = \sigma_i = ((\sigma_i)_{\alpha\beta})$, $i = 1, 2, 3$, and we take $B = \varepsilon = (\varepsilon^{\alpha\beta})$. Here $\alpha = -, +$ are two-spinor indices and the quantity $\varepsilon^{\alpha\beta}$ is the invariant antisymmetric tensor with $\varepsilon^{-+} = +1$. We use $\varepsilon^{\alpha\beta}$ to raise indices, with the convention $\psi^\alpha = \varepsilon^{\alpha\beta} \psi_\beta$. Indices will be lowered using $\varepsilon_{\alpha\beta}$, $\varepsilon_{-+} = +1$, with the convention $\psi_\alpha = \psi^\beta \varepsilon_{\beta\alpha}$. If we define $J_i = -\frac{1}{2} \varepsilon_{ijk} M_{jk}$, then the $\mathcal{N} = 1$ super-Poincaré algebra can be rewritten as

$$[J_i, J_j] = \varepsilon_{ijk} J_k, \tag{5.11a}$$

$$[J_i, P_j] = \varepsilon_{ijk} P_k, \tag{5.11b}$$

$$[J_i, Q_\alpha] = \frac{i}{2} (\sigma_i)_{\alpha\beta} Q_\beta, \tag{5.11c}$$

$$[Q_\alpha, Q_\beta] = 2i (\sigma_i \varepsilon)_{\alpha\beta} P_i. \tag{5.11d}$$

Writing the general element of the algebra as $X = \varphi^i J_i + x^i P_i + \theta^\alpha Q_\alpha$ we obtain a real form by imposing reality conditions $(\varphi^i)^\diamond = \varphi^i$, $(x^i)^\diamond = x^i$ and $(\theta^\alpha)^\diamond B^{\alpha\beta} = \theta^\beta$. Exponentiating the algebra gives the super-Poincaré group, $S\Pi$, from which we form the coset space $S\Pi/SO(3)$, where $SO(3)$ is the rotation group generated by the J_i . Following the method discussed in [9] we consider a coset representative

$$L(x^i, \theta^\alpha) = \exp(x^i P_i + \theta^\alpha Q_\alpha), \quad (5.12)$$

so that (x^i, θ^α) are coordinates on the coset space. We hence have $S\Pi/SO(3) = \mathbb{R}^{3|2}$ where reality is defined with respect to pseudo-conjugation as given above.

The left action of $S\Pi$ on the coset representative induces a transformation on the coordinates $(x^i, \theta^\alpha) \rightarrow (x^i + \delta x^i, \theta^\alpha + \delta \theta^\alpha)$. Using this we can find the differential operator representation of the generators of the superalgebra. In particular we have,

$$Q_\alpha = -\partial_\alpha + i(\sigma_i \varepsilon)_{\alpha\beta} \theta^\beta \partial_i. \quad (5.13)$$

An invariant vielbein (E^i, E^α) and spin-connection Ω^i on $\mathbb{R}^{3|2}$ can be constructed from the coset representative as

$$L^{-1} dL = E^i P_i + E^\alpha Q_\alpha + \Omega^i J_i. \quad (5.14)$$

We find that $\Omega^i = 0$, and so the inverse vielbein determines the covariant derivatives, which turn out to be

$$D_i = \partial_i, \quad (5.15a)$$

$$D_\alpha = \partial_\alpha + i(\sigma_i \varepsilon)_{\alpha\beta} \theta^\beta \partial_i. \quad (5.15b)$$

For an even superscalar field $\Phi(x, \theta)$, satisfying $\Phi^\diamond = \Phi$, let us consider the action

$$I = \int d^3x d\theta^- d\theta^+ \left(\frac{1}{2} D^\alpha \Phi D_\alpha \Phi - U(\Phi) \right). \quad (5.16)$$

It is easily seen that $[Q_\alpha, D_\beta] = 0$, from which it follows that this action will be invariant under supersymmetry transformations $\delta\Phi = \beta^\alpha Q_\alpha \Phi$. We can expand Φ in component fields as

$$\Phi(x, \theta) = A(x) + \theta^\alpha \psi_\alpha(x) + \frac{1}{2} \theta^\alpha \theta_\alpha F(x). \quad (5.17)$$

The condition $\Phi^\diamond = \Phi$ yields $A = A^\diamond$, $F = F^\diamond$ and $\psi_\alpha = (B^{-1})_{\alpha\beta} (\psi_\beta)^\diamond$. Hence we see that ψ is a graded Majorana spinor.

The action I can be rewritten in terms of the component fields. Upon elimination of the auxiliary field F via its equations of motion, and integrating out the θ coordinates, I becomes

$$I = \int d^3x \left((\partial A)^2 - \frac{1}{4} U'(A)^2 + i\psi^\alpha (\sigma^i)_{\alpha\beta} \partial_i \psi_\beta + \frac{1}{2} U''(A) \psi^\alpha \psi_\alpha \right). \quad (5.18)$$

This is the action for a real scalar field coupled to a graded Majorana spinor in three-dimensional Euclidean space. For an example of a supersymmetric action involving Dirac spinors in this signature, see [10]. Note that, as $C\gamma^\mu$ is symmetric in this signature, a supersymmetric action containing a symplectic action of the form of equation (4.16) does not exist.

6. Conclusions and outlook

We have seen how the classification of possible reality conditions on Grassmann odd valued spinors should be extended by what we call a graded Majorana condition. In contrast to the symplectic Majorana condition which, in order to be imposed, requires an even number

of spinor fields, the graded Majorana condition can be imposed on a single spinor. In fact, as we showed in section 4.3 the graded Majorana condition imposes the same number of constraints on a spinor as does a standard Majorana condition.

In order to illustrate the use of graded Majorana spinors in supersymmetric field theories we constructed an action involving such spinors in the case of three-dimensional Euclidean space. In globally curved space an example of the use of graded Majorana spinors is obtained by considering field theories on the supersphere $S^{2|2} = UOSP(1|2)/U(1)$, as investigated in [11]. Graded Majorana spinors could also play an important role in the construction of supergravity theories. In this context, an interesting example of a spacetime where no standard Majorana spinors exist is 11-dimensional Euclidean space. It will be very interesting to investigate whether the existence of graded Majorana spinors may account for a physically sensible supergravity theory in this spacetime.

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